Theorem: The following is an example of a family of intervals over the real number line $A_n = \{x \in \mathbb{R} : 0 \leq x \leq c \in \mathbb{R}\}$ where $c = \frac{1}{2}(k-n+1+|k-n+1|) \in \mathbb{N}$ for some natural numbers k, n and real number $\epsilon > 0$. If such a family of intervals exist then $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n = \{0\}$.

Proof: We will assume the family of intervals A_n exists. From there we will prove the subset part and then the intersection part separately.

To prove a set A_{n+1} is a subset of A_n for all n, we need to show that if $x \in A_{n+1}$ then $x \in A_n$ for all n.

So pick an arbitrary n. If $x \in A_{n+1}$, then by definition:

$$0 \leq x \leq \tfrac{1}{2}(k-(n+1)+1|k-(n+1)+1|): \ (*)$$

And if $x \in A_n$, then by definition:

$$0 \leq x \leq \tfrac{1}{2}(k-n+1+|k-n+1|): \ (**)$$

We will cover the positive and the non positive case. If $k-n \leq 0$ then $\frac{1}{2}(k-n+|k-n|) = 0$ by the definition of the absolute value. Which means x must be 0 by (*), therefore $x \in A_n$ by (**).

If $k-n > 0$ consider that $(k-n) < (k-n+1)$ and since we are strictly positive $|k-n| = k-n$ and $|k-n+1| = k-n+1$. So after factoring we now have (fractions are allowed since we are dealing with the reals):

$$(k-n) = \tfrac{1}{2}(k-n+|k-n|) < \tfrac{1}{2}(k-n+1+|k-n+1|) = (k-n+1).$$

By (*):

$$0 \leq x \leq \tfrac{1}{2}(k-n+|k-n|) < \tfrac{1}{2}(k-n+1+|k-n+1|)$$

Therefore $x \in A_n$ by (**). Since $x \in A_{n+1}$ implies $x \in A_n$ this proves that $A_{n+1} \subset A_n$ and since n was chosen arbitrarily this holds for all n. The first property is proven.

I want to prove that for some value of n, $\forall m \geq n : k-m+1+|k-m+1| = 0$. Choose n such that $n = k+1$. That means for an $m \geq n$ chosen arbitrarily $m \geq k+1$ Rearrange: $0 \geq k-m+1$: (***).

Proof is continued on next page:

Taking the absolute value: $|k-m+1|=0$ or $|k-m+1|=-(k-m+1)$. Add $k-m+1$ to both sides of (***) and multiply by $\frac{1}{2}$:

$$\tfrac{1}{2}(|k-m+1|+k-m+1)=0.$$

By: (**) $0 \leq x \leq \frac{1}{2}(|k-m+1|+k-m+1)=0.$

Therefore, by the definition of a closed interval, since m was chosen arbitrarily

$$\forall m \geq n, A_m = [0,0] = \{0\} :(****)$$

We will use this to prove the second part of the theorem.

The definition of $\bigcap_{n=1}^{\infty} A_n$: $\{x|(\forall n)(x \in A_n\}$. We can separate A_n into two halves, where $n \geq k+1$ and $n < k+1$. By (**) 0 is a closed lower bound of the interval $\forall n < k+1$ so : $0 \in A_n$ in this case is clear.

By (****), $\forall m \geq n : 0 \in A_m$. Since there are no elements in these sets other than 0 that lets us conclude: $\bigcap_{n=1}^{\infty} A_n = \{0\}$. This proves the second property of the theorem and thus the proof is complete.